Monodromy of Compositions of Toroidal Belyĭ Maps

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Pomona Research in Mathematics Experience

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Table of Contents (Summary)

Background

Elliptic Curves

Elliptic Curve

An **elliptic curve** $E(\mathbb{C})$ is the set of all points (x, y) satisfying a nonsingular equation of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

for coefficients $a_1, a_2, a_3, a_4, a_6 \in \mathbb{C}$.

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for coefficients $a_1, a_2, a_3, a_4, a_6 \in \mathbb{C}$.

Note

Every elliptic curve $E(\mathbb{C})$ is a torus $T^2(\mathbb{R})$.

Toroidal Belyĭ Maps

Belyĭ Map

A **Belyĭ map** $\gamma : X \to \mathbb{P}^1(\mathbb{C})$ is a mapping of a Riemann surface to a Riemann sphere with three branch points $\{0, 1, \infty\}$.

Toroidal Belyĭ Maps

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A **Belyĭ pair** (X, γ) is composed of the Riemann surface and its corresponding Belyĭ map.

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Belyĭ Pair

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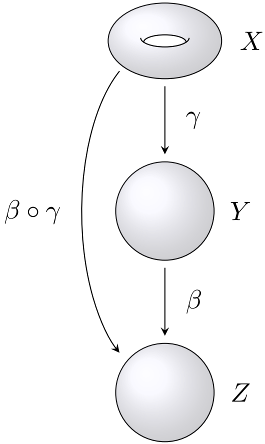
Toroidal Belyĭ Map

A **Toroidal Belyĭ map** is a mapping $\gamma : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ from an elliptic curve E to a Riemann sphere. A **Toroidal Belyĭ pair** is (E, γ).

Toroidal Belyĭ Maps



Dessin d'Enfants

Dessin d'Enfants

Given a Belyĭ pair (X, γ) we define the sets $B = \gamma^{-1}(\{0\})$ and $W = \gamma^{-1}(\{1\})$. We refer to B as the set of black vertices and W as the set of white vertices. The **Dessin d'Enfant** is the bipartite graph embedded in X with vertices B, W and edges $\gamma^{-1}([0, 1])$.

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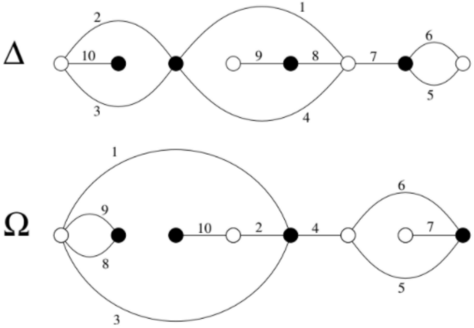
Note

The degree of a Belyĭ map γ is equal to the number of edges in its dessin d'enfant.

Dessin d'Enfants



Δ is the dessin corresponding to the permutation pair

$$[(1,2,3,4)(5,6,7)(8,9),(1,8,4,7)(2,3,10)(5,6)]$$

Ω is the dessin corresponding to the permutation pair

$$[(1,2,3,4)(5,6,7)(8,9),(1,3,8,9)(2,10)(4,5,6)]$$

Degree Sequence

Degree Sequence

Denote the preimages $B = \gamma^{-1}(0)$, $W = \gamma^{-1}(1)$, and $F = \gamma^{-1}(\infty)$ as marked points on the compact connected Riemann surface X. We will define the **Degree Sequence** of γ as the multiset of multisets

$$\mathcal{D} = \Big\{ \{ e_P \,|\, P \in B \}, \{ e_P \,|\, P \in W \}, \{ e_P \,|\, P \in F \} \Big\}.$$

Define $N = \deg(\gamma)$ as the **degree** of the Belyĭ map.

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Example

The degree sequence

$$\mathcal{D} = \{\{4, 3, 2, 1\}, \{4, 3, 2, 1\}, \{10\}\}$$

Monodromy Groups

Monodromy Group

A group of the form $G = \langle \sigma_0, \sigma_1, \sigma_\infty \rangle$ that satisfies these properties is said to be a **monodromy group**.

- Each of the permutations in $\mathcal{D}$ is a product of disjoint cycles with corresponding cycle types.
- G is a transitive subgroup of S_N
- $\sigma_0 \circ \sigma_1 \circ \sigma_\infty = 1$

Wreath Product

Semidirect Product

Given any two groups N, H and a group homomorphism $\varphi : H \to \mathrm{Aut}(N)$ we can construct the **semidirect product** $N \rtimes H$ as follows:

- The underlying set is the product $N \times H$.
- The binary operation $\star$ is defined as

$$(n_1, h_1) \star (n_2, h_2) = (n_1 \varphi(h_1)(n_2), h_1 h_2)$$

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Wreath Product

Let G be a group and $H \leq S_n$ for some non-negative integer n. Then we can form the **wreath product** as

$$G \wr H = G^n \rtimes H$$

where H acts on G^n by permuting the n copies of G.

Our Project

Goal

The monodromy group $\text{Mon}(\beta)$ contains information about the symmetries of a Belyĭ map β. For any Toroidal Belyĭ map γ,

- There is a surjective group homomorphism $\text{Mon}(\beta \circ \gamma) \twoheadrightarrow \text{Mon}(\beta)$.
- The monodromy group $\text{Mon}(\beta \circ \gamma)$ is contained in the wreath product $\text{Mon}(\gamma) \wr \text{Mon}(\beta)$.

Goal

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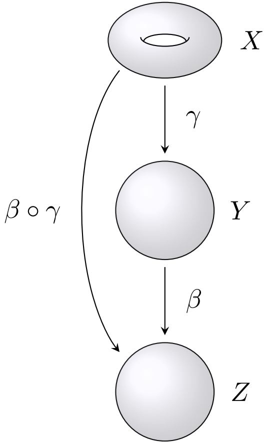
The monodromy group Mon(β) contains information about the symmetries of a Belyĭ map β. For any Toroidal Belyĭ map γ,

- There is a surjective group homomorphism Mon($\beta \circ \gamma$) $\twoheadrightarrow$ Mon(β).
- The monodromy group Mon($\beta \circ \gamma$) is contained in the wreath product Mon(γ) $\wr$ Mon(β).

Goal:

In this project, we study how the three groups Mon(β) and Mon($\beta \circ \gamma$) and Mon(γ) $\wr$ Mon(β) compare as we vary over Dynamical Belyĭ maps β and now **Toroidal Belyĭ** maps γ.

Toroidal Belyĭ Map



We will be working with the composition $\beta \circ \gamma : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, which is a Toroidal Belyĭ Map.

Motivating Question

When is $\text{Mon}(\beta \circ \gamma)$ **equal** to $\text{Mon}(\gamma) \wr \text{Mon}(\beta)$?

Tools

Jacob Bond's Theorems

Corollary (pg. 71)

The monodromy group $\mathsf{Mon}(\beta\gamma)$ of the composition of a dynamical Belyĭ map β and a Belyĭ map γ is isomorphic to a subgroup of the wreath product $\mathsf{Mon}(\gamma) \wr_{E_\beta} \mathsf{Mon}(\beta)$. Moreover, this isomorphism is given by

$$\begin{aligned} \mathsf{Mon}(\beta\gamma) &\to \varphi_\gamma(\pi_1^Z) \leq \mathsf{Mon}(\gamma) \wr_{E_\beta} \mathsf{Mon}(\beta) \\ \rho_{\beta\gamma}(\lambda) &\mapsto (\rho_{\gamma\star}(f_\lambda), \rho_\beta(\lambda)). \end{aligned}$$

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Note

- The wreath product is denoted $\wr_{E_\beta}$ because $\mathrm{Mon}(\beta)$ acts on the set of edges E_β of the dessin for β.
- $\rho_\beta(\lambda)$ denotes the monodromy representation of λ under β.

Jacob Bond's Theorems

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Theorem 4.18 (pg. 76)

Let β be a dynamical Belyĭ map with constellation (τ_0, τ_1), and extending pattern (f_0, f_1). Let

$$\varphi : \begin{array}{ll} g_0 & \mapsto (f_0, \tau_0) \\ g_1 & \mapsto (f_1, \tau_1) \end{array}$$

and $A := \varphi(\text{Ker}\rho_\beta)$. Then for any Belyĭ map γ,

$$\text{Mon}(\beta\gamma) \cong \rho_{\gamma\star}(A) \rtimes \text{Mon}(\beta)$$

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Note

We can view $\text{Ker}\rho_\beta \leq F_2$. If $F_2 = \langle g_0, g_1 \rangle$ then we can construct the above homomorphism φ by defining $\varphi(g_0)$ and $\varphi(g_1)$.

Jacob Bond's Theorems

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Rules for the extending pattern

1. If $p \subseteq \mathcal{R}_{1/2}$, then $p^{\circlearrowleft} \simeq_p 1$.
2. If either $p(0), p(1) \in \overline{\mathbb{H}^+}$ or $p(0), p(1) \in \mathbb{H}^-$ and either $p \subseteq \mathcal{R}_{-1/2}$ or $p \subseteq \mathcal{R}_{3/2}$, then $p \simeq {}_p 1$
3. If $p(0) \in \overline{\mathbb{H}^+}, p(1) \in \mathbb{H}^-$, and $p \subseteq \mathcal{R}_{-1/2}$, then $p^{\circlearrowleft} \simeq_p a$.
4. If $p(0) \in \mathbb{H}^-, p(1) \in \overline{\mathbb{H}^+}$, and $p \subseteq \mathcal{R}_{3/2}$, then $p^{\circlearrowleft} \simeq_p b$.
5. If $p(0) \in \mathbb{H}^-, p(1) \in \overline{\mathbb{H}^+}$, and $p \subseteq \mathcal{R}_{-1/2}$, then $p^{\circlearrowleft} \simeq_p a^{-1}$.
6. If $p(0) \in \overline{\mathbb{H}^+}, p(1) \in \mathbb{H}^-$, and $p \subseteq \mathcal{R}_{3/2}$, then $p^{\circlearrowleft} \simeq_p b^{-1}$.

Note

$$\mathcal{R}_{-1/2} := \mathbb{P}^1(\mathbb{C}) \backslash [0, \infty]$$

$$\mathcal{R}_{1/2} := \mathbb{P}^1(\mathbb{C}) \backslash ([-\infty, 0] \cup [1, \infty])$$

$$\mathcal{R}_{3/2} := \mathbb{P}^1(\mathbb{C}) \backslash [-\infty, 1]$$

Jacob Bond's Theorems

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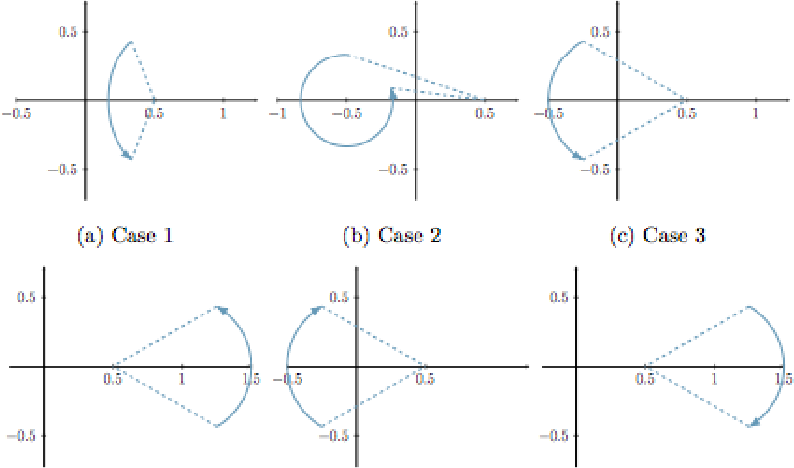
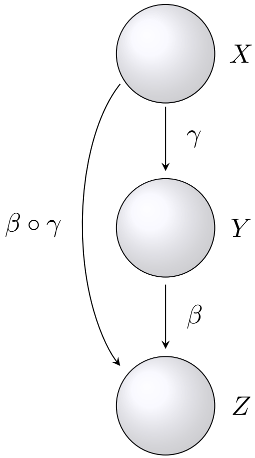


(a) Case 1 (b) Case 2 (c) Case 3

(d) Case 4 (e) Case 5 (f) Case 6

Melanie Wood's Paper

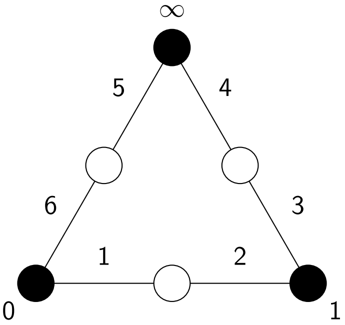


Melanie Wood uses the composition $\beta \circ \gamma : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, mapping a sphere to a sphere to a sphere.

Wood's Paper Cont.

Example 3.8 pg 733

$\xi(t) = 27t^2/(4(t^2 - t + 1)^3)$.
The extending pattern of ξ is shown in the figure below.



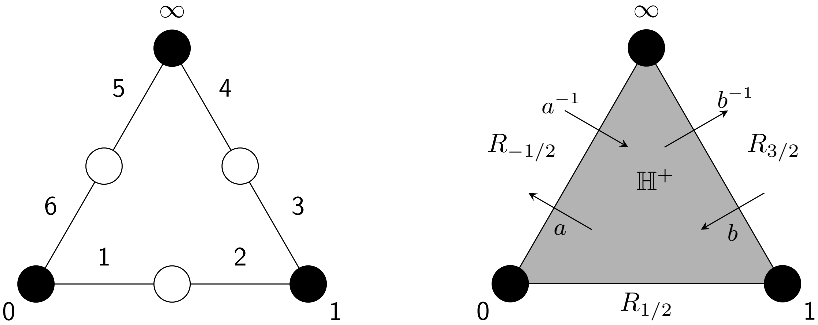
In the notation from Jacob Bond's thesis, for $\gamma = \Delta, \Omega$ and $\beta = \xi$, then we have

$$\tau_0 = (1,6)(2,3)(4,5)$$

$$\tau_1 = (1,2)(3,4)(5,6)$$

Wood's Paper Cont.

Pomona College



Extending Pattern

$$\tau_0 = (1,6)(2,3)(4,5) \quad f_0 = [1, b, 1, b^{-1}a^{-1}, 1, a]$$
$$\tau_1 = (1,2)(3,4)(5,6) \quad f_1 = [1, 1, 1, 1, 1, 1]$$

Wood's Paper Cont.

$$\mathrm{Mon}(\xi) = H = \langle (1,2)(3,4)(5,6), (1,6)(2,3)(4,5) \rangle.$$
$$\mathrm{Mon}(\Delta) = \mathrm{Mon}(\Omega) = A_{10}.$$

Let $n = |A_{10}| = \frac{10!}{2}$.

Wood's Paper Cont.

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$$\mathrm{Mon}(\Delta) = \mathrm{Mon}(\Omega) = A_{10}.$$

Let $n = |A_{10}| = \frac{10!}{2}$.

Then, $A_{10} \wr H$, has order $6n^6$.

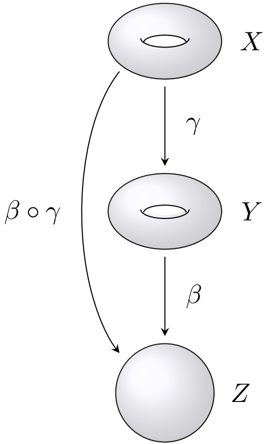
$$|\mathrm{Mon}(\xi \circ \Delta)| = 6n^2,$$

so $\mathrm{Mon}(\xi \circ \Delta) \lneq A_{10} \wr H$, but

$$|\mathrm{Mon}(\xi \circ \Omega)| = 6n^6,$$

so $\mathrm{Mon}(\xi \circ \Omega) = A_{10} \wr H$.

Belyĭ Lattès Maps by Ayberk Zeytin



Ayberk Zeytin uses the composition $\beta \circ \gamma : E(\mathbb{C}) \to E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, mapping a torus to a torus to a sphere.

Belyĭ Lattès Maps by Ayberk Zeytin

Let E be an elliptic curve given by $E : y^2 = x^3 + 1$. Consider the toroidal Belyĭ map

$$\phi : E \to \mathbb{P}^1$$

given by

$$\phi : P = (x, y) \mapsto z = \frac{1 - y}{2}.$$

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$$\phi : P = (x, y) \mapsto z = \frac{1 - y}{2}.$$

Lattès Maps

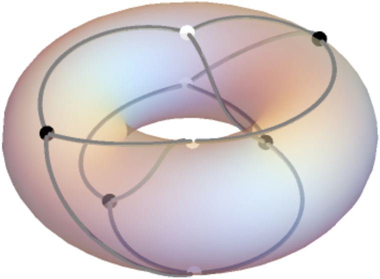
For any positive integer N, the multiplication by N map on E, $[N]$ yields a dynamical Belyĭ map $B_N : \mathbb{P}^1 \to \mathbb{P}^1$ given by $B_N(\phi(P)) = \phi([N])$. Then, B_N has degree N^2 and the B_N are called **Lattès maps**.

Belyĭ Lattès Maps by Ayberk Zeytin

n	B_n	$\mathrm{Mon}(B_n)$	$\mathrm{Mon}(B_n \circ \phi)$
2	$\frac{(z-1)(z+1)^3}{8(z-1/2)^3}$	A_4	A_4
3	$\frac{(z^3+3z^2-6z+1)^3}{27z(z-1)(z^2-z+1)^3}$	He_3 (Heisenberg of order 27)	He_3
4	$\frac{z(z^5+8z^4-32z^3+28z^2-10z+4)^3}{(4z^5-10z^4+28z^3-32z^2+8z+1)^3}$	$(C_4 \times C_4) \rtimes C_3$	$(C_4 \times C_4) \rtimes C_3$

Belyĭ Lattès Maps by Ayberk Zeytin

Case: n=2



(Image from Mathematica code by Elzie, Nishida, and Thomas.)

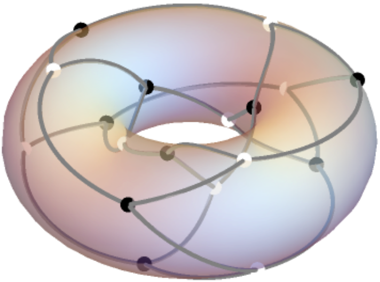
$$\tau_0 = (1,3,4) \quad \tau_1 = (2,4,3)$$

$$f_0 = [1, b, a, a^{-1}] \quad f_1 = [a, b^{-1}, 1, b]$$

Dessin Explorer from REUF and Professor Goins

Belyĭ Lattès Maps by Ayberk Zeytin

Case: n=3



(Image from Mathematica code by Elzie, Nishida, and Thomas.)

$$\tau_0 = (1,7,2)(3,9,4)(5,8,6)$$

$$\tau_1 = (1,2,8)(3,4,7)(5,6,9)$$

$$f_0 = [a^{-1}, a, 1, 1, b, b^{-1}, 1, 1, 1]$$

$$f_1 = [b, 1, a^{-1}, 1, 1, 1, a, b^{-1}, 1]$$

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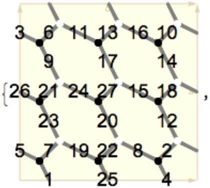
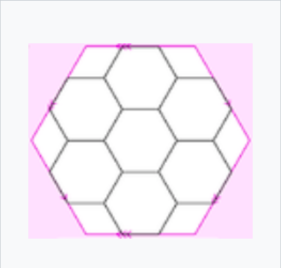
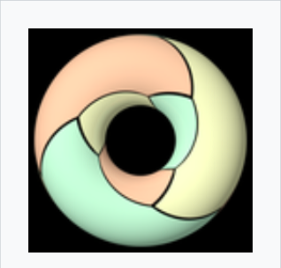
Belyĭ Lattès Maps by Ayberk Zeytin

Case: n=3



Pappus graph: 18 vertices, 27 edges, 9 hexagons





Sagemath

★ A special thanks to Dr. Edray Goins for providing a base code for us to adapt for our own research!

Function of Code

0. Inputs a Belyĭ pair (f, β) where β is written b in our code.

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2. Solve the first order IVP:

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = 2\pi\sqrt{-1}\frac{\beta(x,y) - e}{(\partial\beta/\partial x)(\partial f/\partial y) - (\partial\beta/\partial y)(\partial f/\partial x)} \begin{bmatrix} +\frac{\partial f}{\partial y} \\ -\frac{\partial f}{\partial x} \end{bmatrix}, \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = P_a$$

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   We use Euler's method to do this in Sage.

3. Form a list of endpoints by carrying out step 2 for $a = 1, 2, \ldots, N$ on the interval $0 \leq t \leq 1$ and selecting the endpoint of each path. Do this twice to create 2 lists, one for $e = 0$ and one for $e = 1$.

Sagemath

4. Compare the list of endpoints computed to the list of N points, and take the point P_a from step 1 which is closest to that endpoint. This will help us avoid small rounding errors.

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6. Compute the **monodromy group** of the Belyĭ pair by defining G as the symmetric group of order N and the monodromy group H as the subgroup of G generated by σ_0 and σ_1.

7. Determine isomorphism. Define M as the monodromy group for the Belyĭ pair (f, b) and C as the monodromy group for the Belyĭ pair (f, b^n). Check if $|C| = m^n n$ (the order of the wreath product.)

Results

Results

Theorem

Let γ be a toroidal Belyĭ map and $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be given by $\beta(z) = z^n$ with $n > 1$. Suppose $\text{Mon}(\gamma) = \langle a_\gamma, b_\gamma \rangle$ is abelian, then

$$\text{Mon}(\beta\gamma) \cong \text{Mon}(\gamma) \wr \text{Mon}(\beta) \iff \text{Mon}(\gamma) = \langle b_\gamma \rangle.$$

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Note

Recall, Theorem 4.18 tells us

$$\text{Mon}(\beta\gamma) \cong \text{Mon}(\gamma) \wr \text{Mon}(\beta) \iff \rho_{\gamma^*}(A) \cong (\text{Mon}(\gamma))^n.$$

The latter statement is the approach we take in proving the above theorem.

Proof (sketch)

Goal: Show that $\rho_{\gamma^*}(A) \cong (\text{Mon}(\gamma))^n$ if and only if $\text{Mon}(\gamma) = \langle b_\gamma \rangle$.
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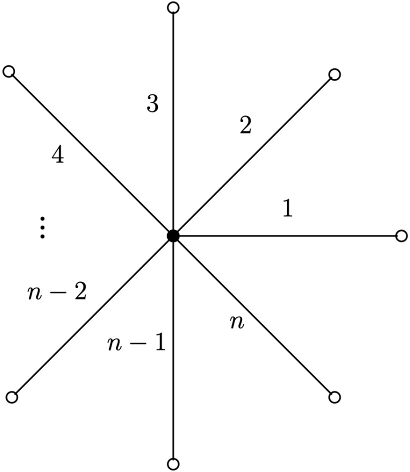
Outline

1. Calculate τ_0, τ_1 and f_0, f_1 for β.
2. Determine generators of $\text{Ker}(\rho_\beta)$.
3. Find generators of $A := \varphi(\text{Ker}(\rho_\beta))$ and subsequently, $\rho_{\gamma^*}(A)$.
4. Show $\text{Mon}(\gamma) = \langle b_\gamma \rangle$ implies $\rho_{\gamma^*}(A) \cong (\text{Mon}(\gamma))^n$.
5. Show $\rho_{\gamma^*}(A) \cong (\text{Mon}(\gamma))^n$ implies $\text{Mon}(\gamma) = \langle b_\gamma \rangle$.

Step 1

1. Calculate τ_0, τ_1 and f_0, f_1 for β.



$$\tau_0 = (1, 2, \ldots, n)$$
$$\tau_1 = id$$
$$f_0 = (1, \ldots, a, \ldots, 1)$$
$$f_1 = (b, 1, \ldots, 1)$$

Step 2

2. Determine generators of Ker(ρ_β)

- Recall that $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ has branch points $0, 1, \infty$ so that $\rho_\beta : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}) \to \text{Mon}(\beta)$.
- The fundamental group $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}) \cong F_2$ where $F_2 = \langle a, b \rangle$.
- $\rho_\beta(a) = \tau_0 = (1, 2, \ldots, n)$ and $\rho_\beta(b) = \tau_1 = id$.
- $\text{Ker}(\rho_\beta) = \langle a^n, b, a^i b a^{-i} \rangle$ for $i \in \{\pm 1, \ldots, \pm\lfloor \frac{n}{2} \rfloor\}$

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Sanity Check:

$$F_2/\langle a^n, b, a^i b a^{-i}\rangle = \{\overline{1}, \overline{a}, \ldots, \overline{a^{n-1}}\} \cong C_n$$

Step 3

3. Find generators of $A := \varphi(\text{Ker}(\rho_\beta))$ and subsequently, $\rho_{\gamma^*}(A)$

Recall: $\varphi(a) = [f_0, \tau_0]$ and $\varphi(b) = [f_1, \tau_1]$

- $\varphi(a) = [(1, \ldots, a, \ldots, 1); \tau_0]$
- $\varphi(b) = [(b, 1, \ldots, 1); id]$

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Example Calculation:

$$\begin{aligned}
\varphi(a)^2 &= [(1, \dots, a, \dots, 1); \tau_0] \cdot [(1, \dots, a, \dots, 1); \tau_0] \\
&= [(1, \dots, a, \dots, 1) \cdot \tau_0(1, \dots, a, \dots, 1); \tau_0^2] \\
&= [(1, \dots, a, \dots, 1) \cdot (1, \dots, 1, a, \dots, 1); \tau_0^2] \\
&= [(1, \dots, a, a, \dots, 1); \tau_0^2]
\end{aligned}$$

Step 3 (cont.)

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Generators of $A := \varphi(\mathrm{Ker}(\rho_\beta))$

- $\varphi(a^n) = [(a, \dots, a); id]$
- $\varphi(b) = [(b, 1, \dots, 1); id]$
- $\varphi(a^i b a^{-i}) = [(1, \dots, b, \dots, 1); id]$

$$\varphi(\mathrm{Ker}(\rho_\beta)) = \langle [(a, \dots, a); id], [(b, 1, \dots, 1); id], \dots, [(1, \dots, 1, b); id] \rangle$$

$\rho_\gamma = \rho_{\gamma^*} \rtimes id$

$$\rho_{\gamma^*}(A) = \langle (a_\gamma, \dots, a_\gamma), (b_\gamma, 1, \dots, 1), \dots, (1, \dots, 1, b_\gamma) \rangle$$

Step 4

4. Show Mon$(\gamma) = \langle b_\gamma \rangle$ implies $\rho_{\gamma^*}(A) \cong (\text{Mon}(\gamma))^n$

- $\rho_{\gamma^*}(A) = \langle (a_\gamma, \ldots, a_\gamma), (b_\gamma, \ldots, 1), \ldots, (1, \ldots, b_\gamma) \rangle \leq (\text{Mon}(\gamma))^n$.
- $\langle (b_\gamma, 1, \ldots, 1), \ldots, (1, \ldots, 1, b_\gamma) \rangle \cong (\langle b_\gamma \rangle)^n \leq \rho_{\gamma^*}(A)$
- Since Mon$(\gamma) = \langle b_\gamma \rangle$, $(\langle b_\gamma \rangle)^n = (\text{Mon}(\gamma))^n$
- $(\text{Mon}(\gamma))^n \leq \rho_{\gamma^*}(A) \leq (\text{Mon}(\gamma))^n$ implies $\rho_{\gamma^*}(A) \cong (\text{Mon}(\gamma))^n$.

Step 5

5. Show $\rho_{\gamma^*}(A) \cong (\mathrm{Mon}(\gamma))^n$ implies $\mathrm{Mon}(\gamma) = \langle b_\gamma \rangle$

- $(a_\gamma, 1, \ldots, 1) \in \rho_{\gamma^*}(A)$
- There exists $\ell, k_1, k_2 \in \mathbb{Z}$ such that

$$a_\gamma = b_\gamma^{k_1} a_\gamma^\ell \text{ and } 1 = b_\gamma^{k_2} a_\gamma^\ell$$

- Then $a_\gamma^\ell = b_\gamma^{-k_2}$ so that $a_\gamma = b_\gamma^{k_1 - k_2}$
- $a_\gamma \in \langle b_\gamma \rangle$ implies $\mathrm{Mon}(\gamma) = \langle b_\gamma \rangle$

Other dynamical Belyĭ maps

We can prove analogous results for other dynamical Belyĭ maps:

i	β	Extending Pattern	Generators
1	$-\frac{27}{4}(t^3-t^2)$	$\tau_0=(12)\quad f_0=[a,1,b]$ $\tau_1=(23)\quad f_1=[1,1,1]$	$[a^{-2},b^{-1},b^{-1}],[1,1,1],[b^{-1},a^{-2},b^{-1}],$ $[ab^{-1}a^{-1},b^{-1},ba^{-2}b^{-1}],$ $[a^{-1},ab^{-1},ba^{-1}b^{-1}]$
2	$-2t^3+3t^2$	$\tau_0=(12)\quad f_0=[a,1,1]$ $\tau_1=(23)\quad f_1=[1,b,1]$	$[a^{-1},a^{-1},1],[1,b^{-1},b^{-1}],[ab^{-1}a^{-1},1,b^{-1}],$ $[a^{-1},1,a^{-1}],[1,a^{-1},a^{-1}],$ $[1,ba^{-1},1],[ab^{-1}a^{-1},a^{-1},1]$
3	$\frac{t^3+3t^2}{4}$	$\tau_0=(23)\quad f_0=[1,a,1]$ $\tau_1=(12)\quad f_1=[1,1,b]$	$[1,a^{-1},a^{-1}],[1,1,b^{-2}],[a,ab^{-2}a^{-1},1],$ $[a^{-1},1,ba^{-1}b^{-1}],[a^{-1},aba^{-1}b^{-1}a^{-1},1],$ $[aba^{-1},b^{-1}a^{-1},b],[ab^{-1}a^{-1},b^{-1}a^{-1},b]$
4	$\frac{27t^2(t-1)}{(3t-1)^3}$	$\tau_0=(23)\quad f_0=[b,a,1]$ $\tau_1=(12)\quad f_1=[b^{-1}a^{-1},1,1]$	$[b^{-2},a^{-1},a^{-1}],[ab,ab,1],[ba,1,ab],$ $[b^{-1}a^{-1}b,b^{-2},a^{-1}],[a^{-1},a^{-1},b^{-2}],$ $[b^{-1},a^{-2},b^{-1}],[b^{-1},ba^{-1},a]$
5	$\frac{t^2(t-1)}{(t-\frac{4}{3})^3}$	$\tau_0=(12)\quad f_0=[a,1,b]$ $\tau_1=(23)\quad f_1=[b^{-1}a^{-1},1,1]$	$[a^{-1},a^{-1},b^{-2}],[b^{-1}a^{-1}b,b^{-2},a^{-1}],$ $[abab,1,1],[1,abab,1],$ $[ab^{-2}a^{-1},b^{-1}a^{-1}b,ba^{-1}b^{-1}],$ $[b^{-2}a^{-1},b^2,b^{-1}a^{-1}b^{-1}],[b^{-2}a^{-1},b^2,a]$

Other dynamical Belyĭ maps

Sufficient conditions for $\mathrm{Mon}(\beta_i\gamma) \cong \mathrm{Mon}(\gamma) \wr \mathrm{Mon}(\beta_i)$

β_1: $\mathrm{Mon}(\gamma) = \langle a_\gamma^2 \rangle$ or $a_\gamma = 1$ (so that $\mathrm{Mon}(\gamma) = \langle b_\gamma \rangle$)

β_2: $\mathrm{Mon}(\gamma) = \langle a_\gamma^2 \rangle$ or $\mathrm{Mon}(\gamma) = \langle b_\gamma^2 \rangle$

β_3: $\mathrm{Mon}(\gamma) = \langle a_\gamma^2 \rangle$ or $\mathrm{Mon}(\gamma) = \langle b_\gamma^2 \rangle$

β_4: $\mathrm{Mon}(\gamma) = \langle c_\gamma^2 \rangle$

β_5: $\mathrm{Mon}(\gamma) = \langle c_\gamma^2 \rangle$

Further Research

- Investigating case where Mon(γ) is non-abelian
- Considering other compositions like $E(\mathbb{C}) \to E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ or involving surfaces of genus > 1

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Thank you for watching!

Questions?